Eulerian and Lagrangian Structure Function's Scaling Exponents in Turbulent Channel Flow

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Z. Naturforsch. **61a,** 624 – 628 (2006); received October 20, 2006

The relation between Eulerian structure function's scaling exponents and Lagrangian ones in turbulent channel flows is explored both theoretically and numerically. A nonlinear parametric transformation between Eulerian structure function's scaling exponents and Lagrangian ones is derived, following Landau and Novikov's frame work. This relation is then compared to some known experimental and numerical results, but mainly to our DNS (direct numerical simulation) results of a fully developed channel flow with $Re_\tau = 100$. The scaling exponents are evaluated in terms of the ESS (extended self-similarity) method, since the Reynolds number is too low to make the standard scaling laws applicable. The agreement between theory and simulation is satisfactory.

Key words: Turbulent Channel Flow; Intermittency; Scaling Exponents.

1. Introduction

Intermittency is a basic feature of fully developed turbulence [1]. For Eulerian turbulence, intermittency provides corrections to Kolmogorov's scaling law [2], which depict a similar relationship among statistical properties of fluctuations at different small scales. The Kolmogorov predictions are based upon the assumptions that the statistical properties of the velocity field are locally homogeneous and isotropic, and that there exists a constant-energy cascade from large to small scales.

For a homogeneous and isotropic turbulence at a sufficiently high Reynolds number, Kolmogorov (1941) presented that, when scale r (modulus of $\boldsymbol{r}$ lies in the inertial range, the Eulerian velocity structure function has a rather simple form [1]:

$$\langle \Delta U^q(r)\rangle = \langle |U(x+r)-U(x)|^q\rangle \sim r^{\zeta_\mathrm{E}(q)}, \quad (1)$$

where q is the order of moment. For a nonintermittent constant dissipation, the scaling exponent $\zeta_\mathrm{E}(q) = q/3$ [2]. For intermittent turbulence, $\zeta_\mathrm{E}(q)$ is a nonlinear function of q; only the third moment has no intermittency correction: $\zeta_\mathrm{E}(3) = 1$. The exponents $\zeta_\mathrm{E}(q)$ have been estimated experimentally for many years, for different types of flows [3 – 6], and are now considered as rather stable and almost universal until moments of order 7 or 8.

The intermittency can also be investigated in a Lagrangian framework [7]. The knowledge of Lagrangian statistics in a fully developed turbulent flow is also a key ingredient for the development of Lagrangian stochastic models in such diverse contexts as turbulent combustion, pollutant dispersion, cloud formation and industrial mixing. Given the importance of this problem, there are comparatively few experimental studies of turbulent Lagrangian statistics due to the difficulty of tracing the particles in an experiment. However, a recent breakthrough has been made by La Porta et al. [8] tracking three-dimensional fluid particles trajectories with a relatively high resolution, which enables the detailed study of Lagrangian statistical properties, while in the same time, numerical simulations, such as large-eddy simulation (LES) and direct numerical simulation (DNS) have also been applied to investigate Lagrangian statistics in homogeneous and isotropic turbulent flows, and also channel flows, and its important contribution has been recently reviewed by Yeung [9].

In a Lagrangian framework, let $V(y_0,t)$ denote the Lagrangian velocity at the time t of the fluid particle originating from y_0 at the time $t = 0$, the flow is described by the velocity $V(y_0,t)$. It is well recognized that Lagrangian velocity increments $\Delta V(\tau) = |V(y_0,t+\tau) - V(y_0,t)|$, where statistical homogeneity is assumed, are quasi-Gaussian for time lags τ of the order T^{L} (T^{L} is the Lagrangian integral time scale), but become increasingly intermittent at higher frequencies [10]. For the temporal increments of the Lagrangian velocity, Landau and Lifshitz [11] proposed a 1/2 law in 1944. Novikov [7] generalized $\Delta V(\tau)$ with a Lagrangian intermittency framework:

$$\langle \Delta V^q(\tau) \rangle \sim \tau^{\zeta_{\mathrm{L}}(q)}. \tag{2}$$

Neglecting intermittency: $\zeta_{\mathrm{L}}(q) = q/2$. In this framework, the third-order moment for the Eulerian velocity is analogous to the second-order moment for the Lagrangian velocity: in case of intermittency, $\zeta_{\mathrm{L}}(q)$ is nonlinear and convex, and the nonintermittent function is valid only for $q = 2$: $\zeta_{\mathrm{L}}(2) = 1$.

Since experimental and numerical estimates are available for both scaling exponents $\zeta_{\mathrm{E}}(q)$ and $\zeta_{\mathrm{L}}(q)$, it is now possible to test some theoretical hypotheses leading to direct or parametric links between these functions. The relation between Eulerian structure functions and Lagrangian ones has been recently considered by Schmitt [12] on theoretical grounds. The aim of our paper is to explore the relation between Eulerian structure function's scaling exponents and Lagrangian ones using DNS data. In Section 2 we briefly discuss the establishment of theoretical relations between $\zeta_{\mathrm{E}}(q)$ and $\zeta_{\mathrm{L}}(q)$, and in Section 3 we discuss the numerical investigations. The conclusion is in Section 4.

2. Theoretical Relations between $\zeta_{\mathrm{E}}(q)$ and $\zeta_{\mathrm{L}}(q)$

Let us first recall that in the multifractal framework one may characterize the velocity differences through the singularities h and their codimension $c(h)$ [13]:

$$\Delta U(r) \sim r^h, \quad p\left(\Delta U(r)\right) \sim r^{c(h)}. \tag{3}$$

Here h is in the range $(0,1)$ [6], which implies that the velocity derivatives are singular, and the codimension is used for convenience, instead of the more frequently used dimension $f(h) = d - c(h)$, where d is the dimension of the space ($d =$ 1, 2, or 3), and $p\left(\Delta U(r)\right)$ is the probability density of velocity fluctuations. The moments are written as

$$\begin{aligned} \langle \Delta U^q(r) \rangle &= \int \Delta U^q(r) p\left(\Delta U(r)\right) \mathrm{d}\Delta U(r) \\ &\sim \int_0^1 r^{qh+c(h)} \mathrm{d}h \sim r^{\zeta_{\mathrm{E}}(q)}. \end{aligned} \tag{4}$$

In the limit $r \to 0$, the integral in (4) is dominated by the power law with the smallest exponent, and a steepest descent argument gives the classical Legendre transform between $\zeta_{\mathrm{E}}(q)$ and the codimension function [1, 6, 13]:

$$\zeta_{\mathrm{E}}(q) = \min_h \{qh + c(h)\}. \tag{5}$$

This can also be written in the following way, emphasizing the one-to-one relation between orders of moment q and singularities h:

$$\begin{aligned} &\zeta_{\mathrm{E}}(q) = qh + c(h), \\ &q = -c'(h). \end{aligned} \tag{6}$$

Obviously, $c(h)$ is assumed to be concave, i. e., $c''(h) > 0$, and there exists an h in the range $(0,1)$ at which the minimum of $qh + c(h)$ must be attained.

Different hypotheses can be invoked to relate Eulerian and Lagrangian statistics and obtain some relations for Eulerian and Lagrangian scaling exponents; see [12] for more details. Some researchers consider the velocity advecting Lagrangian trajectories as a superposition of different velocity contributions coming from different eddies. In a time lag τ the contribution from eddies smaller than a given scale r, are uncorrelated, and one may then write [10]

$$\Delta V(\tau) \sim \Delta U(r). \tag{7}$$

Taking into account the intermittency for the Eulerian velocity, one can write $\Delta U(r) \sim r^h$. Since we have also $\Delta U(r) \sim r/\tau$, this gives [10]

$$\tau \sim r^{1-h}. \tag{8}$$

This corresponds to taking into account the intermittency of the dissipation, and to assuming a local relation between space and time, influenced by intermittency.

To relate the Eulerian and Lagrangian scaling exponents, let's consider the statistical relation (7) and the time-scale relation (8). Using (8) to introduce the local time-space relation inside the integral, one obtains the q-th moments of the velocity increments:

$$\langle \Delta V^q(\tau)\rangle \sim \langle \Delta U^q(r)\rangle \sim \int_0^1 \tau^{\frac{qh+c(h)}{1-h}}\,\mathrm{d}h. \tag{9}$$

In the limit $\tau \to 0$, steepest descent argument again gives the exponent of Lagrangian structure function as a Legendre transform:

$$\zeta_{\mathrm{L}}(q) = \min_h \left(\frac{qh+c(h)}{1-h}\right) \equiv \min_h G(h). \tag{10}$$

Introducing the singularity h_0 that minimizes $G(h)$ in the range $(0,1)$, i.e., $G'(h_0)=0$, it can be derived that $G''(h_0) = c''(h_0)/(1-h_0)$, which satisfies $G''(h_0) > 0$ with the assumption of concavity of $c(h)$. Then we have

$$\zeta_{\mathrm{L}}(q) = \frac{qh_0+c(h_0)}{1-h_0}. \tag{11}$$

Solving $G'(h_0) = 0$, and after introducing the moment of order q_0 associated to h_0 through the Legendre transform (6), i.e., $\zeta_{\mathrm{E}}(q_0) = q_0h_0 + c(h_0)$, and $q_0 = -c'(h_0)$, one obtains a simple relation between q and q_0 (thus providing a unique value of q_0 and hence of h_0, for a given value of q):

$$q_0 - \zeta_{\mathrm{E}}(q_0) = q. \tag{12}$$

Substitution of (12) together with $\zeta_{\mathrm{E}}(q_0) = q_0h_0 + c(h_0)$ and $q_0 = -c'(h_0)$ into (11) gives finally the following parametric relation between Lagrangian and Eulerian scaling exponents:

$$\begin{aligned} \zeta_{\mathrm{L}}(q) &= \zeta_{\mathrm{E}}(q_0), \\ q &= q_0 - \zeta_{\mathrm{E}}(q_0). \end{aligned} \tag{13}$$

The Euler-Lagrange relation given in (13) is obviously nonlinear, which, to some extent, is the implication of the complexity of turbulence structure functions. For the known lognormal model,

$$\zeta_{\mathrm{E}}(q) = aq - bq^2 \tag{14}$$

with $a = (2+\mu)/6$ and $b = \mu/18$, where $\mu = 0.23$ is the intermittency parameter. From (13), we obtain

$$\zeta_{\mathrm{L}}(q) = (a-1+\sqrt{(1-a)^2+4qb}-2bq)/2b. \tag{15}$$

This lognormal model (14) and (15) will be used to compare our DNS results in a turbulent channel flow and experimental estimations published in recent papers [3–6, 14].

3. Numerical Evaluation of the Relations between $\zeta_{\mathrm{E}}(q)$ and $\zeta_{\mathrm{L}}(q)$

In this section, the relations (13)–(15), derived in the preceding section, are tested using our DNS results in a channel flow for a relatively low Reynolds number $Re_\tau = 100$, where $Re_\tau = u_\tau\delta/\nu$, and u_τ, δ and ν represent the friction velocity, half width of the channel and kinematic viscosity, respectively. The dependence of scaling exponents on the distance from the wall (or the starting points in the Lagrangian exponents) is also briefly investigated. A pseudo-spectral numerical scheme is used in our DNS. The domain size in the streamwise, wall-normal, and spanwise directions is $1880\cdot 200\cdot 630$ in wall units, respectively. The corresponding grid number in each direction is $96\cdot 65\cdot 64$. The velocity fields are computed and stored at the interval $\Delta t^+ = \Delta t u_\tau^2/\nu = 0.2$ for a period of 3000 in wall units. The fluid particle velocity along a particle trajectory is computed by employing the third-order Hermite polynomials in the homogeneous directions and a Chebyshev polynomial in the wall-normal direction. Periodic boundary conditions are imposed on the streamwise and spanwise directions to obtain the velocity of the fluid particles.

For a turbulence at low or moderate Reynolds numbers, there is no wide enough inertial range to determine the scaling exponents. To extend the applicability of the scaling law, Benzi et al. [3] developed an ESS (extended self-similarity) model. Using ESS, an accurate estimate for the scaling exponents [$\zeta_{\mathrm{E}}(q)$ and $\zeta_{\mathrm{L}}(q)$] could be obtained at low Reynolds number. We shall consider only the streamwise scaling exponents $\zeta_{\mathrm{E}}(q)$ and $\zeta_{\mathrm{L}}(q)$. The differences of scaling exponents between different velocity exponents shall be addressed elsewhere.

Eulerian structure function's scaling exponent in the streamwise direction $\zeta_{\mathrm{E}}(q)$ is plotted in Figure 1. Av.Exp stands for the average of four experimental estimations for $\zeta_{\mathrm{E}}(q)$ in homogeneous and isotropic fully developed turbulence [3–6]. It is clear from this figure that further deviation from $q/3$ is observed when approaching the wall [the smaller $y^+(=yu_\tau/\nu)$], which means a stronger intermittency near the wall than in the central of the channel. The increase in intermittency toward the wall has also been observed by Antonia et al. [15]. It is also clearly shown that $\zeta_{\mathrm{E}}(q)$ in the

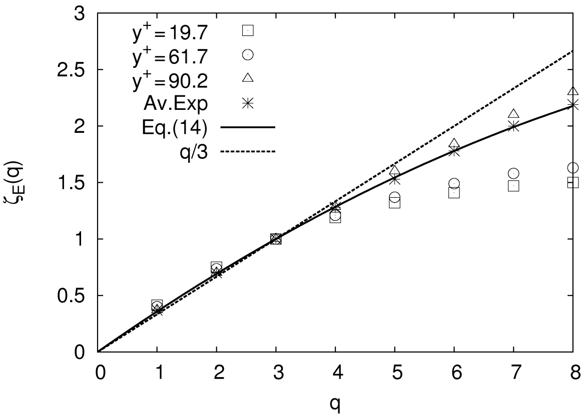

Fig. 1. Comparison of Eulerian structure function's scaling exponent $\zeta_{\rm E}(q)$ at $y^+ = 19.7$, 61.7 and 90.2 in the streamwise direction. Av.Exp stands for the average of four experimental estimations for $\zeta_{\rm E}(q)$ in homogeneous and isotropic fully developed turbulence [3 – 6].

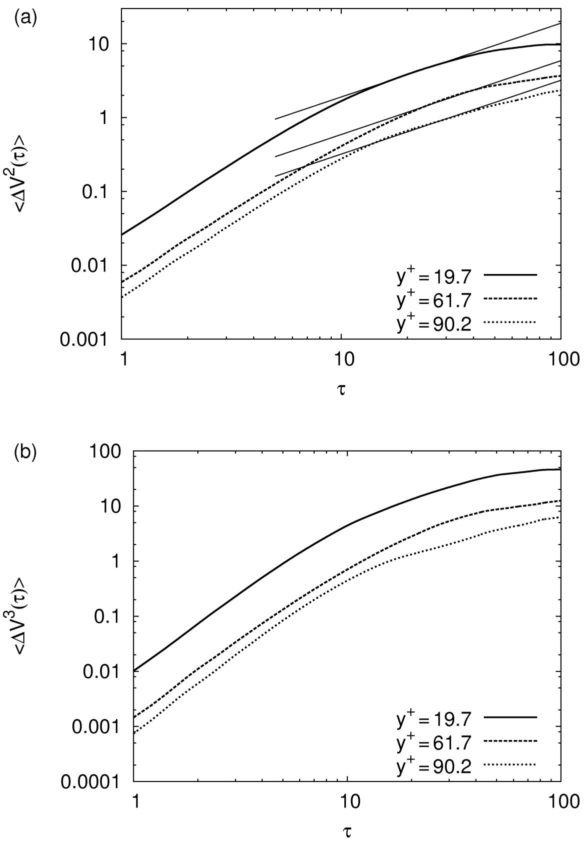

Fig. 2. (a) $\langle \Delta V^2(\tau)\rangle$ vs. τ for particles released from $y^+ = 19.7$, 61.7 and 90.2. (b) Same as (a) for the third-order structure function, namely $\langle \Delta V^3(\tau)\rangle$.

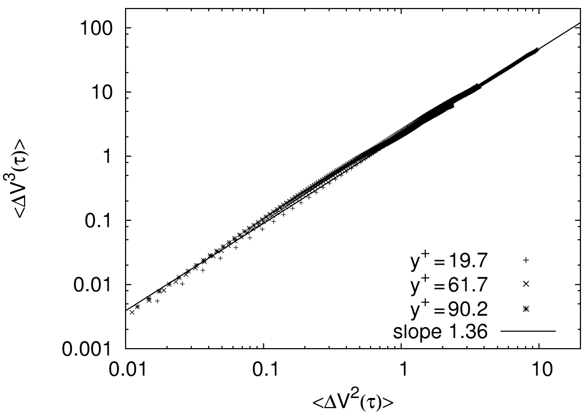

Fig. 3. $\langle \Delta V^3(\tau)\rangle$ vs. $\langle \Delta V^2(\tau)\rangle$ for particles released from $y^+ = 19.7$, 61.7 and 90.2.

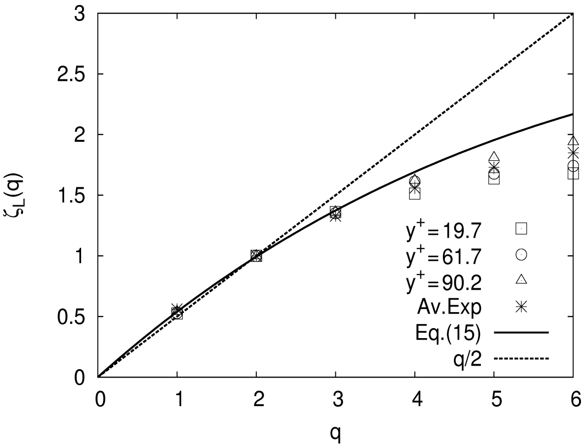

Fig. 4. Comparison of the Lagrangian structure function's scaling exponent $\zeta_{\rm L}(q)$ for particles released from $y^+ = 19.7$, 61.7 and 90.2.

central range ($y^+ = 90.2$) agrees well with the experimental estimation and prediction with the lognormal models (14).

Figure 2 shows the second and third moments of the Lagrangian velocity increment ($\langle \Delta V^2(\tau)\rangle$ and $\langle \Delta V^3(\tau)\rangle$) for particles released from $y^+ = 19.7$, 61.7 and 90.2, respectively. From Fig. 2a, a rather small inertial range (indicated by the thin solid lines with slope 1) is observed for all three initial locations, and the same result can be seen from Fig. 2b as well. From Fig. 2b, using the same range of scales as indicated in Fig. 2a, a value of $\zeta_{\rm L}(3)$ can be estimated; this yields $\zeta_{\rm L}(3) \approx 1.36$ for $y^+ = 90.2$. As we have mentioned earlier in this section, an ESS method may be applied to evaluate the relative scaling exponents. Following the ESS method, Fig. 3 shows $\langle \Delta V^3(\tau)\rangle$

versus $\langle \Delta V^2(\tau) \rangle$ for particles released from three different heights. A strikingly wider scaling range is observed in Fig. 3, and it is not difficult to determine $\zeta_{\mathrm{L}}(3) = 1.36$ for $y^+ = 90.2$. It is also clearly shown that the scaling exponents are slightly becoming smaller when the starting point approaches the solid wall. In the same way the exponents $\zeta_{\mathrm{L}}(q)$ for $q = 1, 4, 5, 6$ can be estimated. The corresponding results are plotted in Figure 4. Here Av.Exp is the average of 4 experiments published by Mordant et al. [14]. The agreement with (15) is good for low moments. For larger moments ($q \geq 4$), it is clear from Fig. 4 that the intermittency increases with decrease of the initial distance from the wall, which is also confirmed through the observation of $\zeta_{\mathrm{E}}(q)$. Besides, $\zeta_{\mathrm{L}}(q)$ in the central range ($y^+ = 90.2$) almost coincides with experimental estimate. Finally, it is clearly seen that (15) is close to the experimental and DNS result at $y^+ = 90.2$, which has infered that the argument in Section 2, leading to (13), is correct.

4. Conclusion

The relation between Eulerian structure function's scaling exponents and Lagrangian ones has been explored both theoretically and numerically. An explicit parametric formula, relating $\zeta_{\mathrm{L}}(q)$ and $\zeta_{\mathrm{E}}(q)$, has been derived and tested mainly through our own DNS results in a channel flow of $Re_\tau = 100$. The intermittency is found to increase in the near-wall region from investigation of both the Eulerian and Lagrangian scaling behavior, which is attributed to the presence of strong mean shear and organized motions in the near-wall region. It is found that the present ESS evaluation of $\zeta_{\mathrm{E}}(q)$ and $\zeta_{\mathrm{L}}(q)$ for the structure function of the streamwise velocity component in the channel centre is close to experimental estimates and a lognormal fit [(14) and (15)].

Acknowledgements

This research is supported by the Hori Information Science Promotion Foundation and Shanghai Key Discipline Project under Grant No. Y0103. Z.-M. Lu and Y.-L. Liu are also supported by NSFC with Grant No. 10472063 and 10442003.

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